ABSOLUTELY SUMMING OPERATORS REVISITED: NEW DIRECTIONS IN THE NONLINEAR THEORY

A.T. BERNARDINO, D. PELLEGRINO*, J.B. SEOANE-SEPÚLVEDA**, AND M.L.V. SOUZA

Abstract. In the last decades many authors have become interested in the study of multilinear and polynomial generalizations of families of operator ideals (such as, for instance, the ideal of absolutely summing operators). However, these generalizations must keep the essence of the given operator ideal and there seems not to be a universal method to achieve this. The main task of this paper is to discuss, study, and introduce multilinear and polynomial extensions of the aforementioned operator ideals taking into account the already existing methods of evaluating the adequacy of such generalizations. Besides this subject's intrinsic mathematical interest, the main motivation is our belief (based on facts that shall be presented) that some of the already existing approaches are not adequate.

1. Introduction and historical background

A well-known fact from an undergraduate Analysis course states that, in $\mathbb{R}$, a series converges absolutely if and only if it is unconditionally convergent; this result was proved by J.P.G.L. Dirichlet in 1829. For infinite-dimensional Banach spaces the situation is quite different: on the one hand for ℓ_p spaces with $1 < p < \infty$, for example, it is quite easy to construct an unconditionally convergent series which fails to be absolutely convergent. On the other hand, for ℓ_1 and some other Banach spaces the answer to this problem is far from being straightforward. The special case of ℓ_1 was solved in 1947 by M.S. Macphail [42] through a very elaborated construction.

The question of whether every infinite-dimensional Banach space has an unconditionally convergent series which fails to be absolutely convergent was raised by Banach [4, p. 40] (see also Problem 122 in the Scottish Book [46], proposed by S. Mazur and W. Orlicz). In 1950, A. Dvoretzky and C.A. Rogers [31] solved this question in the positive:

Theorem (Dvoretzky-Rogers, 1950). The unconditionally convergent series and absolutely summing convergent series coincide in a Banach space E if and only if $\dim E = \infty$.

The above result encouraged the curiosity of the genius of A. Grothendieck, who rapidly presented a different proof of this result in his Ph.D. dissertation [35]. Grothendieck's famous Résumé [34] (see also [24] for a modern and thorough study) and [35] are, essentially, the beginning of the theory of absolutely summing operators. More precisely, in view of Dvoretzky-Rogers' striking result, the idea of investigating linear operators that transform unconditionally convergent series into absolutely convergent series seemed natural and was the birth of the notion of absolutely summing operators

2010 *Mathematics Subject Classification.* 46G25, 47H60, 47B10.
Key words and phrases. Absolutely summing operators, coherent ideals, compatible ideals, Banach polynomial ideals.
*Supported by CNPq Grant 301237/2009-3.
**Supported by the Spanish Ministry of Science and Innovation, grant MTM2009-07848.

(a linear operator $u : E \to F$ is absolutely summing if $\sum u(x_j)$ is absolutely convergent whenever $\sum x_j$ is unconditionally convergent). Soon after, Grothendieck proved a quite surprising result asserting that every continuous linear operator from ℓ_1 to ℓ_2 (or to any Hilbert space) is absolutely summing (this kind of result is now called a *coincidence theorem*). This result is a consequence of an intriguing inequality which Grothendieck himself called "the fundamental theorem of the metric theory of tensor products". Grothendieck's inequality has important applications ([3, 32]) and still has some hidden mysteries such as the precise value of Grothendieck's constant. For a recent work on the estimates for Grothendieck's constant we refer to [12].

The modern notion of absolutely $(p;q)$-summing operators was introduced in the 1960's by A. Pietsch [63] and B. Mitiagin and A. Pełczyński [49]. Besides its intrinsic mathematical interest and deep mathematical motivation, it has shown to be a very important tool in general Banach space theory. For instance, and just to cite some, using the theory of absolutely summing operators one can show that every normalized unconditional basis of ℓ_1 is equivalent to the unit vector basis of ℓ_1 and also that, for $1 < p < \infty$, there is a normalized unconditional basis of ℓ_p which is not equivalent to the unit vector basis of ℓ_p.

Throughout this paper $\mathbb{N}$ represents the set of all positive integers and $\mathbb{N}_m := \{1, ..., m\}$. Also, $E, E_1, \ldots, E_n, F, G, G_1, ..., G_n, H$ will stand for Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the topological dual of E is represented by E^* and B_{E^*} denotes its closed unit ball. The symbol $W(B_{E^*})$ represents the probability measures in the Borel sets of B_{E^*} with the weak-star topology. We will denote the space of all continuous n-linear operators from $E_1 \times \cdots \times E_n$ into F by $\mathcal{L}(E_1, \ldots, E_n; F)$ or $\mathcal{L}_n(E_1, \ldots, E_n; F)$. Also, we recall that an n-homogeneous polynomial $P : E \to F$ is a map so that $P(x) = \check{P}(x, \ldots, x)$, where $\check{P}$ represents the unique symmetric n-linear map associated to P. The corresponding space (endowed with the sup norm) is represented by $\mathcal{P}(^nE; F)$. For the theory of polynomials and multilinear operators acting on Banach spaces we refer to [29, 50].

For $0 < p < \infty$, the space of all sequences $(x_j)_{j=1}^{\infty}$ in E such that $(\varphi(x_j))_{j=1}^{\infty} \in \ell_p$, for every $\varphi \in E^*$ is denoted by $\ell_p^w(E)$. When endowed with the norm (p-norm if $0 < p < 1$)

$$\left\|(x_j)_{j=1}^{\infty}\right\|_{w,p} := \sup\{(\sum_{j=1}^{\infty} |\varphi(x_j)|^p)^{1/p} : \varphi \in B_{E^*}\},$$

the space $\ell_p^w(E)$ is complete. We recall that if $0 < q \leq p < \infty$ a continuous linear operator $u : E \to F$ is absolutely $(p;q)$-summing if $(u(x_j))_{j=1}^{\infty} \in \ell_p(F)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_q^w(E)$. In this case we write $u \in \Pi_{(p;q)}(E;F)$. For $p = q = 1$ this notion coincides with the concept of absolutely summing operator. For classical results on absolutely summing operators we refer to [23, 48, 74] and references therein (recent results can also be checked in [11, 39, 53]). The concept of absolutely summing operators has some natural linear extensions such as the notions of mixing $(p;q)$-summing operators (due to A. Pietsch and B. Maurey) and $(p;q;r)$-summing operators (due to A. Pietsch). It is worth mentioning that these concepts were not just constructed to simply generalize the notion of absolutely $(p;q)$-summing operators; these notions have their particular reasons to be investigated (see [67, p. 359]).

In the 1980's, Pietsch [66] suggested a multilinear approach to the theory of absolutely summing operators and, more generally, to the theory of operator ideals. Since then, several authors were attracted by the subject and also non-multilinear approaches have appeared (see [16, 17, 37, 43, 45, 57]). The adequate way of lifting the notion of a given operator ideal to the multilinear and polynomial settings is a delicate matter. For example, in the case of the ideal of absolutely summing linear operators, there are several different approaches to the polynomial and multilinear contexts (see [57, 61] and references therein). The abstract notions of (global) holomorphy types (see [7, 52]), coherent and compatible ideals (see [14]) shed some light on what kind of approach is more adequate.

Recently, in 2003, the notion of multiple summing multilinear operators (and polynomials) was introduced (see [44, 62]) but, as a matter of fact, the origin of this notion dates back to [6, 41, 71]. Several indicators from the theory of summing operators and from the theory of (multi-) ideals show that this is one of the most adequate approaches to the nonlinear theory of absolutely summing operators. For results on multiple summing multilinear operators we refer to [10, 21, 60, 62, 68, 69].

Notwithstanding the quick success of the theory of multiple summing multilinear operators, some recent papers related to multilinear summability seem to have overlooked its advantages. More precisely, the multilinear notions of mixing summing operators and absolutely $(p;q;r)$-summing multilinear operators were introduced following a different perspective (see [1, 72]). The point is that these approaches do not carry out the essence of the respective linear concepts and this lack is clearly corroborated by the notions of coherence, compatibility and holomorphy types.

In this paper we present multilinear and polynomial notions of absolutely $(p;q;r)$-summing operators and mixing summing operators which follow the philosophy of the idea of multiple summability. Among other results, the adequacy of our approach is evaluated by proving that our new definitions provide coherent sequences, compatible and also (global) holomorphy types.

Below we recall the notions of mixing summing operators and absolutely $(p;q;r)$-summing operators.

1.1. **Mixing summing operators.** Let $0 < p \leq s \leq \infty$ and r such that $\frac{1}{r} + \frac{1}{s} = \frac{1}{p}$. A sequence $(x_i)_{i=1}^{\infty}$ in E is $(s;p)$-mixed summable if

$$x_i = \tau_i y_i$$

with $(\tau_i)_{i=1}^{\infty} \in \ell_r$ and $(y_i)_{i=1}^{\infty} \in \ell_s^w(E)$.

In this case, consider

$$\|(x_i)_{i=1}^{\infty}\|_{mx(s,p)} := \inf \left\{ \|(\tau_i)_{i=1}^{\infty}\|_r \, \|(y_i)_{i=1}^{\infty}\|_{w,s} \right\},$$

where the infimum is taken over all possible representations of $(x_i)_{i=1}^{\infty}$ in the above form. The space of all $(s;p)$-mixed summable sequences in E is represented by $\ell_{(s,p)}^{mx}(E)$. It is not difficult to prove that $\ell_{(s,p)}^{mx}(E)$ is a complete normed (p-normed if $0 < p < 1$) space.

It is immediate that, for $0 < p \leq s \leq \infty$, one always has

- $\ell_p(E) \subset \ell_{(s,p)}^{mx}(E) \subset \ell_p^w(E)$ with

$$\left\|(z_j)_{j=1}^{\infty}\right\|_{w,p} \leq \left\|(z_j)_{j=1}^{\infty}\right\|_{mx(s,p)} \leq \left\|(z_j)_{j=1}^{\infty}\right\|_{p}, \tag{1.1}$$

- $\ell_p^w(E) = \ell_{(p,p)}^{mx}(E)$ and $\ell_p(E) = \ell_{(\infty,p)}^{mx}(E)$ isometrically.

Let us now recall the linear concept of mixing summing linear operators (see [65]):

Let $0 < p \leq s \leq \infty$. A continuous linear operator $u : E \to F$ is mixing (s,p)-summing ($u \in \Pi_{mx(s,p)}(E;F)$) if there exists a constant $\sigma \geq 0$ such that

$$\left\|(u(x_j))_{j=1}^{m}\right\|_{mx(s,p)} \leq \sigma \left\|(x_j)_{j=1}^{m}\right\|_{w,p} \tag{1.2}$$

for all $x_1, \ldots, x_m \in E$ and $m \in \mathbb{N}$. The infimum of all such constants σ is represented by $\pi_{mx(s,p)}(u)$.

The terminology "mixing" is motivated by the fact that a continuous linear operator $u : E \to F$ is (s,p)-mixing summing precisely when u maps every weakly p-summable sequence $(x_i)_{i=1}^{\infty}$ in E into a sequence which can be written as a product $(\tau_i y_i)_{i=1}^{\infty}$ of an absolutely r-summable scalar sequence $(\tau_i)_{i=1}^{\infty}$ and a weakly s-summable sequence $(y_i)_{i=1}^{\infty}$ in F, where $\frac{1}{s}+\frac{1}{r}=\frac{1}{p}$. Many of the classical results of mixing summing operators are due to B. Maurey [47] and the theory has shown to be sufficiently rich to be investigated by its own (see [19, Section 32]).

1.2. **Absolutely $(p;q;r)$-summing operators.** The concept of absolutely $(p;q;r)$-summing linear operators is due to A. Pietsch [64, 65]. If $0<p,q<\infty$ and $0<r\leq\infty$ and

$$\frac{1}{p}\leq\frac{1}{q}+\frac{1}{r},$$

a continuous linear operator $u:E\to F$ is absolutely $(p;q;r)$-summing $(u\in\Pi_{as(p;q;r)}(E;F))$ if there is a constant $C>0$ such that

$$\left(\sum_{j=1}^{m}\left|\varphi_j\left(u\left(x_j\right)\right)\right|^p\right)^{\frac{1}{p}}\leq C\left\|\left(x_j\right)_{j=1}^m\right\|_{w,q}\left\|\left(\varphi_j\right)_{j=1}^m\right\|_{w,r}\tag{1.3}$$

for all positive integer m, and all $x_1,\ldots,x_m$ in E and $\varphi_1,\ldots,\varphi_m$ in F^*. When $r=\infty$, we recover the classical notion of absolutely $(p;q)$-summing operators. For details we refer to [38, 65, 67].

The space composed by all continuous linear operators from E to F that are absolutely $(p;q;r)$-summing shall be represented by $\Pi_{as(p;q;r)}(E;F)$. The infimum of the constants C satisfying the inequality (1.3) defines a norm (p-norm if $0<p<1$) in $\Pi_{as(p;q;r)}(E;F)$, denoted by $\pi_{(p;q;r)}(u)$. If $r=\infty$ we use the classical notation of absolutely $(p;q)$-summing operators, $\Pi_{(p;q)}(E;F)$ and $\pi_{(p;q)}$ for the norm.

If we allow $\frac{1}{p}>\frac{1}{q}+\frac{1}{r}$ we would have $\Pi_{as(p;q;r)}(E;F)=\{0\}$ (see [27, p. 196]) and, for this reason, we ask for $\frac{1}{p}\leq\frac{1}{q}+\frac{1}{r}$ in the definition above.

1.3. **Operator ideals, multi-ideals and polynomial ideals.** The theory of operator ideals goes back to J.W. Calkin [13], H. Weyl [75] and further work of A. Grothendieck [33]. However, only in the 70's, with A. Pietsch [65], the theory was organized in the modern presentation (see also [25, 36]). For historical details we suggest [67] and for applications we refer to [25].

An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}_1$ of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components

$$\mathcal{I}(E;F):=\mathcal{L}_1(E;F)\cap\mathcal{I}$$

satisfy the following:

(Oa) $\mathcal{I}(E;F)$ is a linear subspace of $\mathcal{L}_1(E;F)$ which contains the finite rank operators.

(Ob) If $u\in\mathcal{I}(E;F)$, $v\in\mathcal{L}_1(G;E)$ and $w\in\mathcal{L}_1(F;H)$, then $w\circ u\circ v\in\mathcal{I}(G;H)$.

The operator ideal is called a normed operator ideal if there is a function $\|\cdot\|_{\mathcal{I}}\colon\mathcal{I}\longrightarrow[0,\infty)$ satisfying

(Ob1) $\|\cdot\|_{\mathcal{I}}$ restricted to $\mathcal{I}(E;F)$ is a norm, for all Banach spaces E, F.

(Ob2) $\|P_1\colon\mathbb{K}\longrightarrow\mathbb{K}:P_1(\lambda)=\lambda\|_{\mathcal{I}}=1$.

(Ob3) If $u\in\mathcal{I}(E;F)$, $v\in\mathcal{L}_1(G;E)$ and $w\in\mathcal{L}_1(F;H)$, then

$$\|w\circ u\circ v\|_{\mathcal{I}}\leq\|w\|\|u\|_{\mathcal{I}}\|v\|.$$

When $\mathcal{I}(E;F)$ with the norm above is always complete, $\mathcal{I}$ is called a Banach operator ideal.

Absolutely summing operators and the two related aforementioned concepts are examples of operator ideals. Other examples include the compact, weakly compact, strictly singular operators, etc.

The notion of multi-ideals is also due to Pietsch [66]. For each positive integer n, let $\mathcal{L}_n$ denote the class of all continuous n-linear operators between Banach spaces. An ideal of multilinear mappings (or multi-ideal) $\mathcal{M}$ is a subclass of the class $\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}_n$ of all continuous multilinear operators between Banach spaces such that for a positive integer n, Banach spaces $E_1, \ldots, E_n$ and F, the components

$$\mathcal{M}_n(E_1, \ldots, E_n; F) := \mathcal{L}_n(E_1, \ldots, E_n; F) \cap \mathcal{M}$$

satisfy:

(Ma) $\mathcal{M}_n(E_1, \ldots, E_n; F)$ is a linear subspace of $\mathcal{L}_n(E_1, \ldots, E_n; F)$ which contains the n-linear mappings of finite type.

(Mb) If $T \in \mathcal{M}_n(E_1, \ldots, E_n; F)$, $u_j \in \mathcal{L}_1(G_j; E_j)$ for $j = 1, \ldots, n$ and $v \in \mathcal{L}_1(F; H)$, then

$$v \circ T \circ (u_1, \ldots, u_n) \in \mathcal{M}_n(G_1, \ldots, G_n; H).$$

Moreover, $\mathcal{M}$ is a (quasi-) normed multi-ideal if there is a function $\|\cdot\|_{\mathcal{M}} \colon \mathcal{M} \longrightarrow [0, \infty)$ satisfying

(Mb1) $\|\cdot\|_{\mathcal{M}}$ restricted to $\mathcal{M}_n(E_1, \ldots, E_n; F)$ is a (quasi-) norm, for all Banach spaces $E_1, \ldots, E_n$ and F.

(Mb2) $\|T_n \colon \mathbb{K}^n \longrightarrow \mathbb{K} : T_n(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \lambda_n\|_{\mathcal{M}} = 1$ for all n,

(Mb3) If $T \in \mathcal{M}_n(E_1, \ldots, E_n; F)$, $u_j \in \mathcal{L}_1(G_j; E_j)$ for $j = 1, \ldots, n$ and $v \in \mathcal{L}_1(F; H)$, then

$$\|v \circ T \circ (u_1, \ldots, u_n)\|_{\mathcal{M}} \leq \|v\| \|T\|_{\mathcal{M}} \|u_1\| \cdots \|u_n\|.$$

When all the components $\mathcal{M}_n(E_1, \ldots, E_n; F)$ are complete under this (quasi-) norm, $\mathcal{M}$ is called a (quasi-) Banach multi-ideal. For a fixed multi-ideal $\mathcal{M}$ and a positive integer n, the class

$$\mathcal{M}_n := \cup_{E_1, \ldots, E_n, F} \mathcal{M}_n \left(E_1, \ldots, E_n; F\right)$$

is called ideal of n-linear mappings.

Similarly, for each positive integer n, let $\mathcal{P}_n$ denote the class of all continuous n-homogeneous polynomials between Banach spaces. A polynomial ideal $\mathcal{Q}$ is a subclass of the class $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ of all continuous homogeneous polynomials between Banach spaces so that for all $n \in \mathbb{N}$ and all Banach spaces E and F, the components

$$\mathcal{Q}_n \left({}^n E; F\right) := \mathcal{P}_n \left({}^n E; F\right) \cap \mathcal{Q}$$

satisfy:

(Pa) $\mathcal{Q}_n \left({}^n E; F\right)$ is a linear subspace of $\mathcal{P}_n \left({}^n E; F\right)$ which contains the finite-type polynomials.

(Pb) If $u \in \mathcal{L}_1 \left(G; E\right)$, $P \in \mathcal{Q}_n \left({}^n E; F\right)$ and $w \in \mathcal{L}_1 \left(F; H\right)$, then

$$w \circ P \circ u \in \mathcal{Q}_n \left({}^n G; H\right).$$

If there exists a map $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \to [0, \infty[$ satisfying

(Pb1) $\|\cdot\|_{\mathcal{Q}}$ restricted to $\mathcal{Q}_n({}^n E; F)$ is a (quasi-) norm for all Banach spaces E and F and all n;

(Pb2) $\|P_n : \mathbb{K} \to \mathbb{K};\ P_n\left(\lambda\right) = \lambda^n\|_{\mathcal{Q}} = 1$ for all n;

(Pb3) If $u \in \mathcal{L}_1(G; E)$, $P \in \mathcal{Q}_n({}^n E; F)$ and $w \in \mathcal{L}_1(F; H)$, then

$$\|w \circ P \circ u\|_{\mathcal{Q}} \leq \|w\| \left\|P\right\|_{\mathcal{Q}} \left\|u\right\|^n,$$

$\mathcal{Q}$ is called (quasi-) normed polynomial ideal. If all components $\mathcal{Q}_n\left({}^nE;F\right)$ are complete, $\left(\mathcal{Q},\|\cdot\|_{\mathcal{Q}}\right)$ is called a (quasi-) Banach ideal of polynomials (or (quasi-) Banach polynomial ideal). For a fixed ideal of polynomials $\mathcal{Q}$ and $n\in\mathbb{N}$, the class

$$\mathcal{Q}_n := \cup_{E,F}\mathcal{Q}_n\left({}^nE;F\right)$$

is called ideal of n-homogeneous polynomials.

A crucial question in the theory of Banach polynomial ideals (and multi-ideals) is the following:

> *Given an operator ideal, is there a natural method to define a related multi-ideal and polynomial ideal without loosing its essence?*

As mentioned before, in general a given operator ideal has several different possible extensions to multi-ideals and polynomial ideals. In an attempt of filtering what approaches are better than others the notions of coherence, compatibility (and in some sense holomorphy types) are quite helpful.

In the last decades several authors have been interested in investigating multilinear and polynomial generalizations of certain operator ideals, such as the ideal of absolutely summing operators. But the search for the correct approach is not an easy task. The generalizations must keep the essence of the given operator ideal and there seems to be no universal receipt for it.

The main goal of this paper is to discuss and introduce multilinear and polynomial extensions of the aforementioned operator ideals (from Subsections 1.1 and 1.2) taking into account the existent methods of evaluating the adequacy of such generalizations. Besides the intrinsic mathematical interest of the subject, the main motivation of this paper is that we believe (based on concrete facts) that the previous approaches were not adequate.

2. Coherence and compatibility

The notions of coherent sequences of ideals of polynomials and compatible ideals of polynomials, which we recall below, are important tools for evaluating polynomial extensions of a given operator ideal. The essence of these concepts rests in the searching of harmony between the levels of homogeneity (n-linearity) of a polynomial ideal and connections (compatibility) with the case of linear operators ($n=1$). In the following if $P\in\mathcal{P}\left({}^nE;F\right)$, then $P_{a^k}\in\mathcal{P}\left({}^{n-k}E;F\right)$ is defined by

$$P_{a^k}(x) := \check{P}(a,\ldots,a,x,\ldots,x).$$

Definition 2.1 (Compatible ideals, [14])**.** *Let $\mathcal{U}$ be a normed ideal of linear operators. A normed ideal of n-homogeneous polynomials $\mathcal{U}_n$ is compatible with $\mathcal{U}$ if there exist positive constants α_1 and α_2 such that for every Banach spaces E and F, the following conditions hold:*

(i) For each $P\in\mathcal{U}_n\left(E;F\right)$ and $a\in E$, $P_{a^{n-1}}$ belongs to $\mathcal{U}\left(E;F\right)$ and

$$\left\|P_{a^{n-1}}\right\|_{\mathcal{U}(E;F)} \leq \alpha_1\left\|P\right\|_{\mathcal{U}_n(E;F)}\left\|a\right\|^{n-1}.$$

(ii) For each $T\in\mathcal{U}\left(E;F\right)$ and $\gamma\in E^*$, $\gamma^{n-1}T$ belongs to $\mathcal{U}_n\left(E;F\right)$ and

$$\left\|\gamma^{n-1}T\right\|_{\mathcal{U}_n(E;F)} \leq \alpha_2\left\|\gamma\right\|^{n-1}\left\|T\right\|_{\mathcal{U}(E;F)}.$$

For the sake of simplicity, we will sometimes write "the sequence $(\mathcal{U}_n)_{n=1}^{\infty}$ is compatible with $\mathcal{U}$" instead of writing "$\mathcal{U}_n$ is compatible with $\mathcal{U}$ for every n". Besides, when we write "the sequence $(\mathcal{U}_n)_{n=1}^{\infty}$ fails to be compatible with $\mathcal{U}$" we are saying that at least for some n, the ideal $\mathcal{U}_n$ is not compatible with $\mathcal{U}$.

Definition 2.2 (Coherent sequence of polynomial ideals [14])**.** *Consider the sequence* $(\mathcal{U}_k)_{k=1}^N$, *where for each* k, $\mathcal{U}_k$ *is an ideal of* k*-homogeneous polynomials and* N *is eventually infinite. The sequence* $(\mathcal{U}_k)_{k=1}^N$ *is a coherent sequence of polynomial ideals if there exist positive constants* β_1 *and* β_2 *such that for every Banach spaces* E *and* F, *the following conditions hold for* $k \in \{1, \ldots, N-1\}$*:*

(*i*) *For each* $P \in \mathcal{U}_{k+1}(E;F)$ *and* $a \in E$, P_a *belongs to* $\mathcal{U}_k(E;F)$ *and*

$$\|P_a\|_{\mathcal{U}_k(E;F)} \leq \beta_1 \|P\|_{\mathcal{U}_{k+1}(E;F)} \|a\| .$$

(*ii*) *For each* $P \in \mathcal{U}_k(E;F)$ *and* $\gamma \in E^*$, γP *belongs to* $\mathcal{U}_{k+1}(E;F)$ *and*

$$\|\gamma P\|_{\mathcal{U}_{k+1}(E;F)} \leq \beta_2 \|\gamma\| \|P\|_{\mathcal{U}_k(E;F)} .$$

3. The first multilinear and polynomial approaches to summability

In 1989, R. Alencar and M.C. Matos [2] explored the following concept of absolutely summing multilinear operators, which was essentially introduced by Pietsch:

Definition 3.1. *Let* $p, p_1, \ldots, p_n \in (0, \infty)$, *with* $\frac{1}{p} \leq \frac{1}{p_1} + \cdots + \frac{1}{p_n}$. *A mapping* $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ *is absolutely* $(p; p_1, \ldots, p_n)$*-summing (or* $(p; p_1, \ldots, p_n)$*-summing) if there exists a* $C \geq 0$ *such that*

$$\left(\sum_{i=1}^m \left\| T(x_i^{(1)}, \ldots, x_i^{(n)}) \right\|^p \right)^{\frac{1}{p}} \leq C \prod_{k=1}^n \left\| \left(x_j^{(k)} \right)_{j=1}^m \right\|_{w,p_k} \tag{3.1}$$

for every $m \in \mathbb{N}$ *and* $x_i^{(k)} \in E_k$, *with* $(i,k) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$. *Analogously an* n*-homogeneous polynomial* $P \in \mathcal{P}(^nE;F)$ *is absolutely* $(p;q)$*-summing if there exists a constant* $C \geq 0$ *such that*

$$\left(\sum_{j=1}^m \|P(x_j)\|^p \right)^{\frac{1}{p}} \leq C \left\| (x_j)_{j=1}^m \right\|_{w,q}^n$$

for all $m \in \mathbb{N}$ *and* $x_j \in E$, *with* $j = 1, \ldots, m$.

The space of all n-linear operators satisfying (3.1) will be denoted by $\mathcal{L}_{as(p;p_1,\ldots,p_n)}(E_1, \ldots, E_n; F)$. When $p_1 = \cdots = p_n = q$, we simply write $\mathcal{L}_{as(p;q)}(E_1, \ldots, E_n; F)$. For $n = 1$ we use the classical notation $\Pi_{(p;q)}$ instead of $\mathcal{L}_{as(p;q)}$. For polynomials we write $\mathcal{P}_{as(p;q)}(^nE;F)$.

For other approaches we mention [9, 18, 21, 28] and references therein. The successful notion of multiple summing multilinear operators will be mentioned in the Section 4.

In the case of mixing summing operators, the multilinear/polynomial theory was investigated by C.A. Soares in his Ph.D. dissertation [72]. However, the definition considered in [72] is an extension of Definition 3.1 and, as it happens to the concept of absolutely summing multilinear operators, it inherits its weaknesses.

Definition 3.2. *Let* $0 < q \leq s \leq \infty$ *and* $0 < p_1, \ldots, p_n \leq \infty$. *An* n*-linear operator* $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ *is* $(s, q; p_1, \ldots, p_n)$*-mixing summing if there exists a constant* $\sigma \geq 0$ *such that*

$$\left\| \left(T(x_j^{(1)}, \ldots, x_j^{(n)}) \right)_{j=1}^m \right\|_{mx(s,q)} \leq \sigma \prod_{k=1}^n \left\| (x_j^{(k)})_{j=1}^m \right\|_{w,p_k} \tag{3.2}$$

for every $m \in \mathbb{N}$, $x_1^{(1)}, \ldots, x_m^{(1)} \in E_1, \ldots, x_1^{(n)}, \ldots, x_m^{(n)} \in E_n$. *Analogously* $P \in \mathcal{P}(^nE;F)$ *is mixing* $(s,q;p)$*-summing if there exists a constant* $C \geq 0$ *such that*

$$\left\| (P(x_j))_{j=1}^m \right\|_{mx(s,q)} \leq C \left\| (x_j)_{j=1}^m \right\|_{w,p}^n$$

for all $m \in \mathbb{N}$ *and* $x_j \in E$, *with* $j = 1, \ldots, m$.

If $p_1 = \cdots = p_n = p$, the operator T is said $(s, q; p)$-mixing summing.

The following multilinear generalization of $(p; q; r)$-summing operators was recently introduced by D. Achour [1]:

Definition 3.3. *Let* $0 < p, q_1, \ldots, q_n < \infty$ *and* $0 < r \leq \infty$ *with*

$$\frac{1}{p} \leq \frac{1}{q_1} + \cdots + \frac{1}{q_n} + \frac{1}{r}.$$

An n*-linear map* $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ *is absolutely* $(p; q_1, \ldots, q_n; r)$*-summing if there is a* $C \geq 0$ *so that*

$$\left(\sum_{j=1}^{m} \left|\varphi_j \left(T\left(x_j^{(1)}, \ldots, x_j^{(n)}\right)\right)\right|^p\right)^{\frac{1}{p}} \leq C \left\|(\varphi_j)_{j=1}^m\right\|_{w,r} \prod_{i=1}^{n} \left\|\left(x_j^{(i)}\right)_{j=1}^m\right\|_{w,q_i} \tag{3.3}$$

for all $m \in \mathbb{N}$, $\varphi_j \in F^*$ *and* $x_j^{(i)} \in E_i$, *with* $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$. *Analogously an* n*-homogeneous polynomial* $P \in \mathcal{P}(^nE; F)$ *is absolutely* $(p; q; r)$*-summing if there exists a constant* $C \geq 0$ *such that*

$$\left(\sum_{j=1}^{m} \left|\varphi_j \left(P\left(x_j\right)\right)\right|^p\right)^{\frac{1}{p}} \leq C \left\|(\varphi_j)_{j=1}^m\right\|_{w,r} \left\|(x_j)_{j=1}^m\right\|_{w,q}^n$$

for all $m \in \mathbb{N}$, $\varphi_j \in F^*$ *and* $x_j \in E$, *with* $j = 1, \ldots, m$.

We denote the space of all absolutely $(p; q_1, \ldots, q_n; r)$-summing n-linear operators by

$$\mathcal{L}_{as(p;q_1,\ldots,q_n;r)}\left(E_1, \ldots, E_n; F\right).$$

When $q_1 = \cdots = q_n = q$ we just write $\mathcal{L}_{as(p;q;r)}\left(E_1, \ldots, E_n; F\right)$. When $r = \infty$ we recover the notion of absolutely $(p; q_1, \ldots, q_n)$-summing multilinear mappings $\mathcal{L}_{as(p;q_1,\ldots,q_n)}$ due to Alencar and Matos [2]. More precisely,

$$\mathcal{L}_{as(p;q_1,\ldots,q_n;\infty)} = \mathcal{L}_{as(p;q_1,\ldots,q_n)}. \tag{3.4}$$

If $\frac{1}{p} > \frac{1}{q_1} + \cdots + \frac{1}{q_n} + \frac{1}{r}$ and T is absolutely $(p; q_1, \ldots, q_n; r)$-summing, then $T = 0$. It is not difficult to prove that

$$\mathcal{L}_{as(p;q_1,\ldots,q_n)}\left(E_1, \ldots, E_n; F\right) \subset \mathcal{L}_{as(p;q_1,\ldots,q_n;r)}\left(E_1, \ldots, E_n; F\right) \tag{3.5}$$

for all Banach spaces $E_1, \ldots, E_n, F$ and $r > 0$.

3.1. **The lack of coherence and compatibility.** The class of absolutely $(p; q)$-summing n-homogeneous polynomials will be denoted by $\mathcal{P}^n_{as(p;q)}$. As before, the space of all n-homogeneous polynomials $P : E \to F$ in $\mathcal{P}^n_{as(p;q)}$ is represented by $\mathcal{P}_{as(p;q)}\left(^nE; F\right)$. The notions of absolutely $(p; q; r)$-summing polynomials and mixing summing polynomials are denoted in a similar way.

It can be easily seen that $\left(\mathcal{P}^n_{as(p;q)}\right)_{n=1}^{\infty}$ in general fails to be coherent and compatible with $\Pi_{as(p;q)}$. In fact for any positive integer $n \geq 2$ and any real number $1 \leq p \leq 2$ we know that

$$\mathcal{P}_{as(1;1)}\left(^n\ell_p; F\right) = \mathcal{P}\left(^n\ell_p; F\right)$$

for all Banach spaces F. This result is an obvious deviation from the spirit of the linear ideal of absolutely summing operators since

$$\Pi_{as(1;1)}\left(\ell_p;F\right)=\mathcal{L}\left(\ell_p;F\right)$$

if and only if $p=1$ and F is a Hilbert space (see [40]). This situation also proves that $\left(\mathcal{P}^n_{as(1;1)}\right)_{n=1}^{\infty}$ is not coherent or compatible with $\Pi_{as(1;1)}$. We also know that $\left(\mathcal{P}^n_{as(p;q)}\right)_{n=1}^{\infty}$ in general is not a (global) holomorphy type.

Since $\mathcal{P}^n_{as(p;q;\infty)}=\mathcal{P}^n_{as(p;q)}$ and $\mathcal{P}^n_{mxs(\infty;p)}=\mathcal{P}^n_{as(p;p)}$ these deficiencies of $\left(\mathcal{P}^n_{as(1;1)}\right)_{n=1}^{\infty}$ are inherited by the polynomial analogues of the concepts of Subsections 1.1 and 1.2. These deficiencies shall be fixed by the alternative concepts introduced in the next sections.

4. Multiple summing multilinear operators: the "nice prototype"

Multiple $(p;q)$-summing multilinear were introduced in 2003 [44, 62]. The origins of this notion date back to the 1930's with Littlewood's 4/3 inequality [41] which asserts that

$$\left(\sum_{i,j=1}^{N}\left|T(e_i,e_j)\right|^{\frac{4}{3}}\right)^{\frac{3}{4}}\leq\sqrt{2}\left\|T\right\|$$

for every bilinear form $T:\ell_\infty^N\times\ell_\infty^N\to\mathbb{K}$ and every positive integer N. In 1931 H.F. Bohnenblust and E. Hille [6] provided a deep generalization of this result to multilinear mappings: for every positive integer n there is a $C_n>0$ so that

$$\left(\sum_{i_1,\ldots,i_n=1}^{N}\left|T(e_{i_1},\ldots,e_{i_n})\right|^{\frac{2n}{n+1}}\right)^{\frac{n+1}{2n}}\leq C_n\left\|T\right\|$$

for every n-linear mapping $T:\ell_\infty^N\times\cdots\times\ell_\infty^N\to\mathbb{C}$ and every positive integer N. This result has important applications in operator theory in Banach spaces, harmonic analysis, complex analysis and analytic number theory. For recent advances related to the Bohnenblust-Hille inequality we refer to [20, 21, 30, 51, 58].

In his Ph.D. dissertation, D. Pérez-García [59] remarked that the Bohnenblust-Hille inequality can be viewed as a result of the theory of multiple summing operators.

Theorem 4.1 (Bohnenblust-Hille)**.** *If $E_1,\ldots,E_n$ are Banach spaces and $T\in\mathcal{L}(E_1,\ldots,E_n;\mathbb{K})$, then there exists a constant $C_n\geq 0$ such that*

$$\left(\sum_{j_1,\ldots,j_n=1}^{N}\left|T(x^{(1)}_{j_1},\ldots,x^{(n)}_{j_n})\right|^{\frac{2n}{n+1}}\right)^{\frac{n+1}{2n}}\leq C_n\prod_{k=1}^{n}\left\|(x^{(k)}_j)_{j=1}^N\right\|_{w,1}\tag{4.1}$$

for every positive integer N and $x^{(k)}_j\in E_k$, $k=1,\ldots,n$ and $j=1,\ldots,N$.

The inequality above can be regarded as a result in the theory of multiple summing multilinear operators. Recall that for $1\leq q_1,\ldots,q_n\leq p<\infty$, an n-linear operator $T:E_1\times\cdots\times E_n\to F$

is multiple $(p;q_1,\ldots,q_n)$-summing ($T\in\mathcal{L}_{mas(p;q_1,\ldots,q_n)}(E_1,\ldots,E_n;F)$) if there exists $C>0$ such that

$$\left(\sum_{j_1,\ldots,j_n=1}^{\infty}\|T(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\|^p\right)^{1/p}\leq C\prod_{k=1}^{n}\|(x_j^{(k)})_{j=1}^{\infty}\|_{w,q_k} \tag{4.2}$$

for every $(x_j^{(k)})_{j=1}^{\infty}\in\ell_{q_k}^w(E_k)$, $k=1,\ldots,n$.

The infimum of all C's satisfying (4.2), denoted by $\|T\|_{(r;r_1,\ldots,r_n)}$, defines a complete norm if $r\geq 1$ (r-norm, if $r\in(0,1)$) in $\mathcal{L}_{mas(r;r_1,\ldots,r_n)}(E_1,\ldots,E_n;F)$. If $r_1=\cdots=r_n=s$ we just write $(r;s)$, and when $r=s$ we replace $(r;r)$ by r. For $n=1$ this concept also coincides with the classical notion of absolutely summing linear operators and, for this reason, we keep the usual notation $\pi_{(r;s)}(T)$ instead of $\|T\|_{(r;s)}$ for the norm of T. The essence of the notion of multiple summing multilinear operators, for bilinear operators, can also be traced back to [71]. For recent results in the theory of multiple summing operators we refer to [8, 22, 60, 68] and references therein.

5. Multiple $(p;q_1,\ldots,q_n;r)$-summing multilinear operators

In this section we introduce the notion of multiple $(p;q_1,\ldots,q_n;r)$-summing multilinear operators and, as we shall see in the next sections, the polynomial version of this concept is coherent and compatible with the (linear) operator ideal of $(p;q;r)$-summing operators.

Definition 5.1. *Let $m\in\mathbb{N},p,r,q_1,\ldots,q_n\geq 1$ and $E_1,\ldots,E_n,F$ be Banach spaces. A continuous multilinear operator $T:E_1\times\cdots\times E_n\to F$ is multiple $(p;q_1,\ldots,q_n;r)$-summing when*

$$\left(\varphi_{j_1\ldots j_n}\left(T\left(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)}\right)\right)\right)_{j_1,\ldots,j_n\in\mathbb{N}}\in\ell_p\left(\mathbb{N}^n\right)$$

whenever $\left(x_j^{(i)}\right)_{j=1}^{\infty}\in\ell_{q_i}^w\left(E_i\right),i=1,\ldots,n$ *and* $\left(\varphi_{j_1\ldots j_n}\right)_{j_1,\ldots,j_n\in\mathbb{N}}\in\ell_r^w\left(F^*,\mathbb{N}^n\right).$

Sometimes we shall simply write $j\in\mathbb{N}^n$ to denote $j=(j_1,\ldots,j_n)\in\mathbb{N}^n$. The vector space formed by the multiple $(p;q_1,\ldots,q_n;r)$-summing multilinear operators from $E_1\times\cdots\times E_n$ to F shall be represented by $\mathcal{L}_{mas(p;q_1,\ldots,q_n;r)}\left(E_1,\ldots,E_n;F\right)$). When $q_1=\cdots=q_n=q$, we simply write $\mathcal{L}_{mas(p;q;r)}\left(E_1,\ldots,E_n;F\right)$.

As it happens in other similar classes, the class $\mathcal{L}_{mas(p;q_1,\ldots,q_n;r)}\left(E_1,\ldots,E_n;F\right)$ has a characterization by means of inequalities:

Theorem 5.2. *The following assertions are equivalent for $T\in\mathcal{L}\left(E_1,\ldots,E_n;F\right)$:*

(i) $T\in\mathcal{L}_{mas(p;q_1,\ldots,q_n;r)}\left(E_1,\ldots,E_n;F\right)$;

(ii) *There is a $C\geq 0$ such that*

$$\begin{aligned}\left(\sum_{j_1,\ldots,j_n=1}^{\infty}\left|\varphi_{j_1\ldots j_n}\left(T\left(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)}\right)\right)\right|^p\right)^{\frac{1}{p}}\\ \leq C\left\|\left(\varphi_{j_1\ldots j_n}\right)_{j_1,\ldots,j_n\in\mathbb{N}}\right\|_{w,r}\prod_{i=1}^{n}\left\|\left(x_j^{(i)}\right)_{j=1}^{\infty}\right\|_{w,q_i}\end{aligned} \tag{5.1}$$

whenever $\left(x_j^{(i)}\right)_{j=1}^{\infty}\in\ell_{q_i}^w\left(E_i\right),i=1,\ldots,n$ *and* $\left(\varphi_{j_1\ldots j_n}\right)_{j\in\mathbb{N}^n}\in\ell_r^w\left(F^*,\mathbb{N}^n\right)$;

(iii) *There is a* $C \geq 0$ *such that*

$$\left(\sum_{j_1,\ldots,j_n=1}^{m} \left| \varphi_{j_1\ldots j_n} \left(T \left(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)} \right) \right) \right|^p \right)^{\frac{1}{p}}$$
$$\leq C \left\| \left(\varphi_{j_1\ldots j_n} \right)_{j_1,\ldots,j_n \in \mathbb{N}_m} \right\|_{w,r} \prod_{i=1}^{n} \left\| \left(x_j^{(i)} \right)_{j=1}^{m} \right\|_{w,q_i}$$

for all $m \in \mathbb{N}$, $x_1^{(i)}, \ldots, x_m^{(i)} \in E_i, i = 1, \ldots, n$ *and* $\left(\varphi_{j_1\ldots j_n} \right)_{j \in \mathbb{N}_m^n} \in \ell_r^w \left(F^*, \mathbb{N}_m^n \right)$.

The infimum of all C *satisfying (5.1) defines a norm in* $\mathcal{L}_{mas(p;q_1,\ldots,q_n;r)} \left(E_1, \ldots, E_n; F \right)$.

Similarly to (3.5) it can also be proved that

$$\mathcal{L}_{mas(p;q_1,\ldots,q_n)} \subset \mathcal{L}_{mas(p;q_1,\ldots,q_n;r)} \tag{5.2}$$

for all $r > 0$. From Theorem 5.2 we can conclude that if

$$\frac{1}{p} > \frac{1}{q_i} + \frac{1}{r}$$

for some i, then $\mathcal{L}_{mas(p;q_1,\ldots,q_n;r)} \left(E_1, \ldots, E_n; F \right) = \{0\}$. In fact, we first prove that if $T \in \mathcal{L}_{mas(p;q_1,\ldots,q_n;r)} \left(E_1, \ldots, E_n; F \right)$, then, for any $a \in E_1$, the map

$$T_a : E_2 \times \cdots \times E_n \longrightarrow F : T_a \left(x_2, \ldots, x_n \right) = T \left(a, x_2, \ldots, x_n \right) \tag{5.3}$$

is multiple $(p; q_2, \ldots, q_n; r)$-summing and

$$\|T\|_{mas(p;q_2,\ldots,q_n;r)} \leq \|a\| \, \|T\|_{mas(p;q_1,\ldots,q_n;r)} . \tag{5.4}$$

So, if $\frac{1}{p} > \frac{1}{q_i} + \frac{1}{r}$ for some i, then $\mathcal{L}_{mas(p;q_1,\ldots,q_n;r)} \left(E_1, \ldots, E_n; F \right) = \{0\}$. In fact, suppose that $\frac{1}{p} > \frac{1}{q_1} + \frac{1}{r}$. So, using (5.3), we know that if $T \in \mathcal{L}_{mas(p;q_1,\ldots,q_n;r)} \left(E_1, \ldots, E_n; F \right)$ then $T_{a_2,\ldots,a_n} \in \mathcal{L}_{as(p;q_1;r)} \left(E_1; F \right)$ for all $a_2 \in E_2, \ldots, a_n \in E_n$. It follows that $T_{a_2,\ldots,a_n} = 0$ and hence $T = 0$. So, in order to avoid trivialities we shall suppose $\frac{1}{p} \leq \frac{1}{q_i} + \frac{1}{r}$ for all i.

5.1. **Coherence and compatibility .** Standard calculations show that

$$\left(\mathcal{L}_{mas(p;q_1,\ldots,q_n;r)}, \|\cdot\|_{mas(p;q_1,\ldots,q_n;r)} \right)$$

is a Banach multi-ideal. If $\mathcal{M}$ is a (quasi-) normed ideal of multilinear mappings, the class

$$\mathcal{P}_{\mathcal{M}} = \left\{ P \in \mathcal{P}^n ; \check{P} \in \mathcal{M}, n \in \mathbb{N} \right\},$$

with $\|P\|_{\mathcal{P}_{\mathcal{M}}} := \left\| \check{P} \right\|_{\mathcal{M}}$, is a (quasi-) normed ideal of polynomials, called polynomial ideal generated by $\mathcal{M}$. If $\mathcal{M}$ is (quasi-) Banach, then $\mathcal{P}_{\mathcal{M}}$ is (quasi-) Banach (see [7, p. 46]).

Thus, the class

$$\mathcal{P}_{mas(p;q;r)}^n = \left\{ P \in \mathcal{P}^n ; \check{P} \in \mathcal{L}_{mas(p;q;r)}^n \right\},$$

with

$$\|P\|_{\mathcal{P}_{mas(p;q;r)}^n} := \left\| \check{P} \right\|_{mas(p;q;r)},$$

ia a Banach polynomial ideal.

Theorem 5.3. $\left(\mathcal{P}_{mas(p;q;r)}^n, \|.\|_{\mathcal{P}_{mas(p;q;r)}^n} \right)_{n=1}^{\infty}$ *is coherent and, for each fixed* n*, compatible with* $\mathcal{L}_{mas(p;q;r)}$.

Proof. If $P \in \mathcal{P}^n_{mas(p;q;r)}\left({}^nE;F\right)$ and $a \in E$, then $\check{P} \in \mathcal{L}^n_{mas(p;q;r)}\left({}^nE;F\right)$ and, from (5.3) and (5.4), $\check{P}_a \in \mathcal{L}^{n-1}_{mas(p;q;r)}\left({}^{n-1}E;F\right)$. Hence $P_a \in \mathcal{P}^{n-1}_{mas(p;q;r)}\left({}^{n-1}E;F\right)$ with

$$\|P_a\|_{\mathcal{P}^{n-1}_{mas(p;q;r)}} \leq \|a\| \, \|P\|_{\mathcal{P}^n_{mas(p;q;r)}}.$$

Let $\gamma \in E^*$. Note that

$$(\gamma P)^{\vee}(x_1,\ldots,x_{n+1}) = \frac{1}{n+1}\sum_{k=1}^{n+1}\gamma(x_k)\,\check{P}\left(x_1,\overset{[k]}{\ldots},x_{n+1}\right),$$

where $\overset{[k]}{\ldots}$ means that the k-th coordinate is missing.

Let $m \in \mathbb{N}$, $x_j^{(k)} \in E$, with $j = 1,\ldots,m$ and $k = 1,\ldots,n+1$; let $\varphi_{j_1\ldots j_{n+1}} \in F^*$ with $j_1,\ldots,j_{n+1} = 1,\ldots,m$. Using the triangle inequality we have

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left|\varphi_{j_1\ldots j_{n+1}}\left((\gamma P)^{\vee}\left(x_{j_1}^{(1)},\ldots,x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}}\\
&=\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left|\varphi_{j_1\ldots j_{n+1}}\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\gamma\left(x_{j_k}^{(k)}\right)\check{P}\left(x_{j_1}^{(1)},\overset{[k]}{\ldots},x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}}\\
&=\frac{1}{n+1}\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left|\varphi_{j_1\ldots j_{n+1}}\left(\sum_{k=1}^{n+1}\gamma\left(x_{j_k}^{(k)}\right)\check{P}\left(x_{j_1}^{(1)},\overset{[k]}{\ldots},x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}}\\
&=\frac{1}{n+1}\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left|\sum_{k=1}^{n+1}\varphi_{j_1\ldots j_{n+1}}\left(\gamma\left(x_{j_k}^{(k)}\right)\check{P}\left(x_{j_1}^{(1)},\overset{[k]}{\ldots},x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}}\\
&\leq\frac{1}{n+1}\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left(\sum_{k=1}^{n+1}\left|\varphi_{j_1\ldots j_{n+1}}\left(\gamma\left(x_{j_k}^{(k)}\right)\check{P}\left(x_{j_1}^{(1)},\overset{[k]}{\ldots},x_{j_{n+1}}^{(n+1)}\right)\right)\right|\right)^p\right)^{\frac{1}{p}}\\
&=\frac{1}{n+1}\left\|\left(\sum_{k=1}^{n+1}\left|\varphi_{j_1\ldots j_{n+1}}\left(\gamma\left(x_{j_k}^{(k)}\right)\check{P}\left(x_{j_1}^{(1)},\overset{[k]}{\ldots},x_{j_{n+1}}^{(n+1)}\right)\right)\right|\right)_{j_1,\ldots,j_{n+1}=1}^{m}\right\|_p\\
&=(*).
\end{aligned}$$

Thus, from the Minkowski inequality we have

$$
\begin{aligned}
(*) &= \\
&= \frac{1}{n+1}\left\|\sum_{k=1}^{n+1}\left(\left|\varphi_{j_1\ldots j_{n+1}}\left(\gamma\left(x_{j_k}^{(k)}\right)\check{P}\left(x_{j_1}^{(1)},\overset{[k]}{\ldots},x_{j_{n+1}}^{(n+1)}\right)\right)\right|\right)_{j_1,\ldots,j_{n+1}=1}^{m}\right\|_p \qquad (5.5)\\
&\leq \frac{1}{n+1}\sum_{k=1}^{n+1}\left\|\left(\left|\varphi_{j_1\ldots j_{n+1}}\left(\gamma\left(x_{j_k}^{(k)}\right)\check{P}\left(x_{j_1}^{(1)},\overset{[k]}{\ldots},x_{j_{n+1}}^{(n+1)}\right)\right)\right|\right)_{j_1,\ldots,j_{n+1}=1}^{m}\right\|_p\\
&= \frac{1}{n+1}\sum_{k=1}^{n+1}\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left|\varphi_{j_1\ldots j_{n+1}}\left(\gamma\left(x_{j_k}^{(k)}\right)\check{P}\left(x_{j_1}^{(1)},\overset{[k]}{\ldots},x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}}\\
&= \frac{1}{n+1}\left[\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left|\varphi_{j_1\ldots j_{n+1}}\left(\check{P}\left(\gamma\left(x_{j_1}^{(1)}\right)x_{j_2}^{(2)},\ldots,x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}}+\cdots\right.\\
&\left.\cdots+\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left|\varphi_{j_1\ldots j_{n+1}}\left(\check{P}\left(\gamma\left(x_{j_{n+1}}^{(n+1)}\right)x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)}\right)\right)\right|^p\right)^{\frac{1}{p}}\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left|\varphi_{j_1\ldots j_{n+1}}\left((\gamma P)^{\vee}\left(x_{j_1}^{(1)},\ldots,x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}} \qquad (5.6)\\
&\leq \frac{1}{n+1}\left[\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left|\varphi_{j_1\ldots j_{n+1}}\left(\check{P}\left(\gamma\left(x_{j_1}^{(1)}\right)x_{j_2}^{(2)},\ldots,x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}}+\cdots\right.\\
&\left.\cdots+\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left|\varphi_{j_1\ldots j_{n+1}}\left(\check{P}\left(\gamma\left(x_{j_{n+1}}^{(n+1)}\right)x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)}\right)\right)\right|^p\right)^{\frac{1}{p}}\right].
\end{aligned}
$$

Note that each one of the $n+1$ terms of (5.6) can be re-written as

$$
\left(\sum_{j_2=1}^{m^2}\sum_{j_3,\ldots,j_{n+1}=1}^{m}\left|\widetilde{\varphi}_{j_2\ldots j_{n+1}}\left(\check{P}\left(z_{j_2}^{(2)},\ldots,z_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}}
$$

for adequate choices of $\widetilde{\varphi}_{j_2\ldots j_{n+1}}$ and $z_{j_k}^{(k)}$, with $k=2,\ldots,n+1$.

In fact, for

$$
\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left|\varphi_{j_1\ldots j_{n+1}}\left(\check{P}\left(\gamma\left(x_{j_1}^{(1)}\right)x_{j_2}^{(2)},\ldots,x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}},
$$

we choose

$$\begin{cases} z_{j_2}^{(2)} = \gamma\left(x_1^{(1)}\right) x_{j_2}^{(2)} \text{ for all } j_2 = 1, \ldots, m, \\ z_{m+j_2}^{(2)} = \gamma\left(x_2^{(1)}\right) x_{j_2}^{(2)} \text{ for all } j_2 = 1, \ldots, m, \\ \vdots \\ z_{(m-1)m+j_2}^{(2)} = \gamma\left(x_m^{(1)}\right) x_{j_2}^{(2)} \text{ for all } j_2 = 1, \ldots, m, \\ z_{j_i}^{(i)} = x_{j_i}^{(i)} \text{ for all } j_i = 1, \ldots, m, i = 3, \ldots, n+1 \end{cases}$$

and

$$\begin{cases} \widetilde{\varphi}_{j_2,\ldots,j_{n+1}} = \varphi_{1j_2\ldots j_{n+1}} \text{ for all } j_2 = 1, \ldots, m, \\ \widetilde{\varphi}_{m+j_2,\ldots,j_{n+1}} = \varphi_{2j_2\ldots j_{n+1}} \text{ for all } j_2 = 1, \ldots, m, \\ \vdots \\ \widetilde{\varphi}_{(m-1)m+j_2,\ldots,j_{n+1}} = \varphi_{mj_2\ldots j_{n+1}} \text{ for all } j_2 = 1, \ldots, m. \end{cases}$$

For these choices one can check that

$$\begin{aligned} &\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m} \left|\varphi_{j_1\ldots j_{n+1}}\left(\check{P}\left(\gamma\left(x_{j_1}^{(1)}\right) x_{j_2}^{(2)}, \ldots, x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{j_2=1}^{m^2} \sum_{j_3,\ldots,j_{n+1}=1}^{m} \left|\widetilde{\varphi}_{j_2\ldots j_{n+1}}\left(\check{P}\left(z_{j_2}^{(2)}, \ldots, z_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}} \end{aligned}$$

and the other cases are similar. Then

$$\begin{aligned} &\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m} \left|\varphi_{j_1\ldots j_{n+1}}\left(\check{P}\left(\gamma\left(x_{j_1}^{(1)}\right) x_{j_2}^{(2)}, \ldots, x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{j_2,\ldots,j_{n+1}=1}^{m^2,m,\ldots,m} \left|\widetilde{\varphi}_{j_2\ldots j_{n+1}}\left(\check{P}\left(z_{j_2}^{(2)}, \ldots, z_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}} \\ &\leq \left\|\check{P}\right\|_{mas(p;q;r)} \left\|\left(\widetilde{\varphi}_{j_2\ldots j_{n+1}}\right)_{j_2,\ldots,j_{n+1}}^{m^2,m,\ldots,m}\right\|_{w,r} \left\|\left(z_{j_2}^{(2)}\right)_{j_2=1}^{m^2}\right\|_{w,q} \prod_{i=3}^{n+1} \left\|\left(z_{j_i}^{(i)}\right)_{j_i=1}^{m}\right\|_{w,q} \\ &= \left\|\check{P}\right\|_{mas(p;q;r)} \left\|\left(\varphi_{j_1\ldots j_{n+1}}\right)_{j\in\mathbb{N}_m^{n+1}}\right\|_{w,r} \left\|\left(\gamma\left(x_{j_1}^{(1)}\right) x_{j_2}^{(2)}\right)_{j_1,j_2=1}^{m}\right\|_{w,q} \prod_{i=3}^{n+1} \left\|\left(x_j^{(i)}\right)_{j=1}^{m}\right\|_{w,q}. \end{aligned}$$

Since

$$\begin{aligned}
&\left\|\left(\gamma\left(x_{j_1}^{(1)}\right)x_{j_2}^{(2)}\right)_{j_1,j_2=1}^m\right\|_{w,q} \\
&\leq \left\|\left(\gamma\left(x_{j_1}^{(1)}\right)\right)_{j_1=1}^m\right\|_\infty \sup_{\|\varphi\|\leq 1}\left(\sum_{j=1}^m\left|\varphi\left(x_{j_2}^{(2)}\right)\right|^q\right)^{\frac{1}{q}} \\
&\leq \left\|\left(\gamma\left(x_{j_1}^{(1)}\right)\right)_{j_1=1}^m\right\|_q \left\|\left(x_{j_2}^{(2)}\right)_{j_2=1}^m\right\|_{w,q} \\
&\leq \|\gamma\| \left\|\left(x_{j_1}^{(1)}\right)_{j_1=1}^m\right\|_{w,q} \left\|\left(x_{j_2}^{(2)}\right)_{j_2=1}^m\right\|_{w,q},
\end{aligned}$$

we have

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_{n+1}=1}^m\left|\varphi_{j_1\ldots j_{n+1}}\left(\check{P}\left(\gamma\left(x_{j_1}^{(1)}\right)x_{j_2}^{(2)},\ldots,x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}} \\
&\leq \|\gamma\|\left\|\check{P}\right\|_{mas(p;q;r)}\left\|\left(\varphi_{j_1\ldots j_{n+1}}\right)_{j\in\mathbb{N}_m^{n+1}}\right\|_{w,r}\prod_{i=1}^{n+1}\left\|\left(x_j^{(i)}\right)_{j=1}^m\right\|_{w,q}.
\end{aligned}$$

Using the same idea for the other n terms of (5.6), we obtain

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_{n+1}=1}^m\left|\varphi_{j_1\ldots j_{n+1}}\left(\check{P}\left(\gamma\left(x_{j_2}^{(2)}\right)x_{j_1}^{(1)},x_{j_3}^{(3)}\ldots,x_{j_{n+1}}^{(n+1)}\right)\right)\right|^p\right)^{\frac{1}{p}} \\
&\leq \|\gamma\|\left\|\check{P}\right\|_{mas(p;q;r)}\left\|\left(\varphi_{j_1\ldots j_{n+1}}\right)_{j\in\mathbb{N}_m^{n+1}}\right\|_{w,r}\prod_{i=1}^{n+1}\left\|\left(x_j^{(i)}\right)_{j=1}^m\right\|_{w,q},
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_{n+1}=1}^m\left|\varphi_{j_1\ldots j_{n+1}}\left(\check{P}\left(\gamma\left(x_{j_{n+1}}^{(n+1)}\right)x_{j_1}^{(1)},x_{j_2}^{(2)}\ldots,x_{j_n}^{n}\right)\right)\right|^p\right)^{\frac{1}{p}} \\
&\leq \|\gamma\|\left\|\check{P}\right\|_{mas(p;q;r)}\left\|\left(\varphi_{j_1\ldots j_{n+1}}\right)_{j\in\mathbb{N}_m^{n+1}}\right\|_{w,r}\prod_{i=1}^{n+1}\left\|\left(x_j^{(i)}\right)_{j=1}^m\right\|_{w,q}.
\end{aligned}$$

Therefore

$$\left(\sum_{j_1,\ldots,j_{n+1}=1}^{m}\left|\varphi_{j_1\ldots j_{n+1}}\left((\gamma P)^{\vee}\left(x_{j_1}^{(1)},\ldots,x_{j_{n+1}}^{(n+1)}\right)\right)\right|^{p}\right)^{\frac{1}{p}}$$
$$\leq \frac{1}{n+1}\left[\|\gamma\|\left\|\check{P}\right\|_{mas(p;q;r)}\left\|\left(\varphi_{j_1\ldots j_{n+1}}\right)_{j\in\mathbb{N}_m^{n+1}}\right\|_{w,r}\prod_{i=1}^{n+1}\left\|\left(x_j^{(i)}\right)_{j=1}^{m}\right\|_{w,q}+\cdots\right.$$
$$\left.\cdots+\|\gamma\|\left\|\check{P}\right\|_{mas(p;q;r)}\left\|\left(\varphi_{j_1\ldots j_{n+1}}\right)_{j\in\mathbb{N}_m^{n+1}}\right\|_{w,r}\prod_{i=1}^{n+1}\left\|\left(x_j^{(i)}\right)_{j=1}^{m}\right\|_{w,q}\right]$$
$$=\|\gamma\|\left\|\check{P}\right\|_{mas(p;q;r)}\left\|\left(\varphi_{j_1\ldots j_{n+1}}\right)_{j\in\mathbb{N}_m^{n+1}}\right\|_{w,r}\prod_{i=1}^{n+1}\left\|\left(x_j^{(i)}\right)_{j=1}^{m}\right\|_{w,q}.$$

Finally we conclude that γP is multiple $(p;q;r)$-summing and

$$\begin{aligned}\|\gamma P\|_{\mathcal{P}^{n+1}_{mas(p;q;r)}} &\leq \|\gamma\|\left\|\check{P}\right\|_{mas(p;q;r)}\\ &=\|\gamma\|\,\|P\|_{\mathcal{P}^{n}_{mas(p;q;r)}}.\end{aligned}$$

The items (i) and (ii) from Definition 2.1 are obtained in a similar way. □

6. Multiple mixing summing operators

In this section we introduce the notion of multiple mixing summing multilinear operators (and polynomials) which is coherent and compatible with the respective operator ideal. As another indicator that this is a correct approach to nonlinear mixing summability, we prove a quotient theorem for multilinear operators similar to the one for mixing summing linear operators.

Definition 6.1. *Let $0<p_1,\ldots,p_n\leq q\leq s<\infty$. An n-linear operator $A\in\mathcal{L}(E_1,\ldots,E_n;F)$ is multiple $(s,q;p_1,\ldots,p_n)$-mixing summing if there exists a constant $\sigma\geq 0$ such that*

$$\left\|\left(A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right)_{j_1,\ldots,j_n=1}^{m}\right\|_{mx(s,q)}\leq\sigma\prod_{k=1}^{n}\left\|(x_j^{(k)})_{j=1}^{m}\right\|_{w,p_k}\tag{6.1}$$

for every $m\in\mathbb{N}$, $x_1^{(1)},\ldots,x_m^{(1)}\in E_1,\ldots,x_1^{(n)},\ldots,x_m^{(n)}\in E_n$.

In this case we define

$$\|A\|_{mx(s,q;p_1,\ldots,p_n)}=\inf\sigma.$$

If $p_1=\cdots=p_n=p$, we say that A is multiple $(s,q;p)$-mixing summing. The space of all multiple $(s,q;p_1,\ldots,p_n)$-mixing summing is represented by $\Pi_{mx(s,q;p_1,\ldots,p_n)}$.

In order to avoid trivialities in the definition of multiple $(s,q;p_1,\ldots,p_n)$ mixing summing operators, we assume that $p_k\leq q$, for all $k=1,\ldots,n$. In fact, one can check that if $T\in\mathcal{L}(E_1,\ldots,E_n;F)$ is multiple $(s,q;p_1,\ldots,p_n)$ mixing summing and $q<p_k$, for some k, then $T=0$.

The following result, whose proof is standard and we omit, characterizes multiple $(s,q;p_1,\ldots,p_n)$ mixing summing operators as those which take adequate weakly summable sequences into adequate mixed summable sequences:

Proposition 6.2. *Let $0 < p_1, \ldots, p_n \leq q \leq s < \infty$. An operator $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is multiple $(s, q; p_1, \ldots, p_n)$-mixing summing if, and only if,*

$$\left(A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)})\right)_{j_1,\ldots,j_n=1}^{\infty} \in \ell_{(s,q)}^{mx}(F, \mathbb{N}^n)$$

regardless of the choice of $(x_i^{(1)})_{i=1}^{\infty} \in \ell_{p_1}^w(E_1), \ldots, (x_i^{(n)})_{i=1}^{\infty} \in \ell_{p_n}^w(E_n)$.

In fact the proof of the previous proposition also shows that A is multiple $(s, q; p_1, \ldots, p_n)$-mixing summing if, and only if, the n-linear operator

$$\tilde{A}\left((x_i^{(1)})_{i=1}^{\infty}, \ldots, (x_i^{(n)})_{i=1}^{\infty}\right) = \left(A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)})\right)_{j_1,\ldots,j_n=1}^{\infty}$$

belongs to $\mathcal{L}(\ell_{p_1}^w(E_1), \ldots, \ell_{p_n}^w(E_n); \ell_{(s,q)}^{mx}(F, \mathbb{N}^n))$. Moreover

$$\|A\|_{mx(s,q;p_1,\ldots,p_n)} = \left\|\tilde{A}\right\|.$$

The main result of this section (Theorem 6.4) is a consequence of the following powerful characterization of mixed summable sequences due to Maurey [47] (see also [65, 16.4.3]):

Theorem 6.3 (Maurey)**.** *Let $0 < q < s < \infty$. A sequence $(z_j)_{j=1}^{\infty}$ in E is mixed (s, q)-summable if, and only if,*

$$\left(\left(\int_{B_{E^*}} |\langle \varphi, z_j \rangle|^s \, d\mu(\varphi)\right)^{\frac{1}{s}}\right)_{j=1}^{\infty} \in \ell_q \text{ whenever } \mu \in W(B_{E^*}).$$

Besides

$$\left\|(z_j)_{j=1}^{\infty}\right\|_{mx(s,q)} = \sup_{\mu \in W(B_{E^*})} \left(\sum_{j=1}^{\infty} \left(\int_{B_{E^*}} |\langle \varphi, z_j \rangle|^s \, d\mu(\varphi)\right)^{\frac{q}{s}}\right)^{\frac{1}{q}}.$$

The next theorem shows that our concept has a characterization similar to the linear case (see [19]*):*

Theorem 6.4. *Let $0 < p_1, \ldots, p_n \leq q \leq s < \infty$. An operator $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is multiple $(s, q; p_1, \ldots, p_n)$ mixing summing if, and only if, there is a constant $\sigma \geq 0$ such that*

$$\left(\sum_{j_1,\ldots,j_n=1}^{m} \left(\sum_{j=1}^{k} \left|\left\langle \varphi_j, A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)})\right\rangle\right|^s\right)^{\frac{q}{s}}\right)^{\frac{1}{q}} \leq \sigma \prod_{l=1}^{n} \left\|(x_i^{(l)})_{i=1}^{m}\right\|_{w,p_l} \left\|(\varphi_j)_{j=1}^{k}\right\|_s \tag{6.2}$$

for all $k, m \in \mathbb{N}$, $x_i^{(l)} \in E_l$; $i = 1, \ldots, m$, $l = 1, \ldots, n$ and $\varphi_j \in F^*$ with $j = 1, \ldots, k$. Furthermore,

$$\|A\|_{mx(s,q;p_1,\ldots,p_n)} = \inf \sigma.$$

Proof. We split the proof into two cases.

(i) Case $s = q$.

From (6.2) we conclude that

$$\left(\sum_{j_1,\ldots,j_n=1}^{m} \left|\left\langle \varphi, A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)})\right\rangle\right|^q\right)^{\frac{1}{q}} \leq \sigma \prod_{l=1}^{n} \left\|(x_i^{(l)})_{i=1}^{m}\right\|_{w,p_l}$$

for all $\varphi \in B_{F^*}$. Thus

$$\left\| \left(A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)}) \right)_{j_1,\ldots,j_n \in \mathbb{N}_m} \right\|_{w,q} \leq \sigma \prod_{l=1}^{n} \left\| (x_i^{(l)})_{i=1}^m \right\|_{w,p_l} \tag{6.3}$$

and so by Theorem 6.3 and by (6.3) we obtain

$$\begin{aligned}
&\left\| \left(A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)}) \right)_{j_1,\ldots,j_n \in \mathbb{N}_m} \right\|_{mx(q,q)} \\
&= \sup_{\mu \in W(B_{F^*})} \left(\sum_{j_1,\ldots,j_n=1}^{m} \left(\int_{B_{F^*}} \left| \left\langle \varphi, A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)}) \right\rangle \right|^q d\mu(\varphi) \right)^{\frac{q}{q}} \right)^{\frac{1}{q}} \\
&\leq \sup_{\mu \in W(B_{F^*})} \left(\int_{B_{F^*}} \sup_{\psi \in B_{F^*}} \sum_{j_1,\ldots,j_n=1}^{m} \left| \left\langle \psi, A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)}) \right\rangle \right|^q d\mu(\varphi) \right)^{\frac{1}{q}} \\
&= \sup_{\mu \in W(B_{F^*})} \left(\int_{B_{F^*}} \left\| \left(A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)}) \right)_{j_1,\ldots,j_n \in \mathbb{N}_m} \right\|_{w,q}^q d\mu(\varphi) \right)^{\frac{1}{q}} \\
&\leq \left\| \left(A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)}) \right)_{j_1,\ldots,j_n \in \mathbb{N}_m} \right\|_{w,q} \\
&\leq \sigma \prod_{l=1}^{n} \left\| (x_i^{(l)})_{i=1}^m \right\|_{w,p_l}.
\end{aligned}$$

Hence, $A \in \Pi_{mx(q,q;p_1,\ldots,p_n)}(E_1, \ldots, E_n; F)$ and $\|A\|_{mx(q,q;p_1,\ldots,p_n)} \leq \sigma$.

Conversely, suppose that $A \in \Pi_{mx(q,q;p_1,\ldots,p_n)}(E_1, \ldots, E_n; F)$. Given

$$x_1^{(1)}, \ldots, x_m^{(1)} \in E_1, \ldots, x_1^{(n)}, \ldots, x_m^{(n)} \in E_n$$

and $\varphi_1, \ldots, \varphi_k \in F^*$, if

$$A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)}) = \tau_{j_1,\ldots,j_n}.y_{j_1,\ldots,j_n},$$

where $(\tau_{j_1,\ldots,j_n})_{j_1,\ldots,j_n \in \mathbb{N}} \in \ell_\infty$ and $(y_{j_1,\ldots,j_n})_{j_1,\ldots,j_n \in \mathbb{N}} \in \ell_q^w(F; \mathbb{N}^n)$ we have

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_n=1}^{m} \left(\sum_{j=1}^{k} \left| \left\langle \varphi_j, A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)}) \right\rangle \right|^q \right)^{\frac{q}{q}} \right)^{\frac{1}{q}} \\
&= \left(\sum_{j=1}^{k} \left(\|\varphi_j\|^q \sum_{j_1,\ldots,j_n=1}^{m} \left| \left\langle \frac{\varphi_j}{\|\varphi_j\|}, \tau_{j_1,\ldots,j_n} y_{j_1,\ldots,j_n} \right\rangle \right|^q \right) \right)^{\frac{1}{q}} \\
&= \left(\sum_{j=1}^{k} \|\varphi_j\|^q \right)^{\frac{1}{q}} \left(\sum_{j_1,\ldots,j_n=1}^{m} |\tau_{j_1,\ldots,j_n}|^q \left| \left\langle \frac{\varphi_j}{\|\varphi_j\|}, y_{j_1,\ldots,j_n} \right\rangle \right|^q \right)^{\frac{1}{q}} \\
&\leq \left\| (\varphi_j)_{j=1}^k \right\|_q \left\| (\tau_{j_1,\ldots,j_n})_{j \in \mathbb{N}^n} \right\|_\infty \left\| (y_{j_1,\ldots,j_n})_{j \in \mathbb{N}^n} \right\|_{w,q}.
\end{aligned}$$

Taking the infimum in both sides, we obtain

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_n=1}^{m}\left(\sum_{j=1}^{k}\left|\left\langle\varphi_j, A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right\rangle\right|^q\right)^{\frac{q}{q}}\right)^{\frac{1}{q}}\\
&\leq \left\|(\varphi_j)_{j=1}^k\right\|_q \left\|\left(A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right)_{j\in\mathbb{N}_m^n}\right\|_{m,(q,q)}\\
&\leq \left\|(\varphi_j)_{j=1}^k\right\|_q \|A\|_{mx(q,q;p_1,\ldots,p_n)} \prod_{l=1}^{n}\left\|(x_i^{(l)})_{i=1}^m\right\|_{w,p_l}.
\end{aligned}$$

Therefore $\inf\sigma \leq \|A\|_{mx(q,q;p_1,\ldots,p_n)}$ and with the last inequality we obtain

$$\|A\|_{mx(q,q;p_1,\ldots,p_n)} = \inf\sigma.$$

(ii) Case $s > q$.

Let $A \in \Pi_{mx(s,q;p_1,\ldots,p_n)}(E_1,\ldots,E_n;F)$. Given $0 \neq \varphi_1,\ldots,\varphi_k \in F^*$ we define the probability measure

$$\nu = \sum_{j=1}^{k}\nu_j\delta_j, \text{ where } \nu_j = \frac{\|\varphi_j\|^s}{\sum_{j=1}^k\|\varphi_j\|^s}$$

and δ_j is the Dirac measure at the point $\tilde{\varphi}_j = \frac{\varphi_j}{\|\varphi_j\|}$.

For $x_1^{(1)},\ldots,x_m^{(1)} \in E_1,\ldots, x_1^{(n)},\ldots,x_m^{(n)} \in E_n$, note that

$$\begin{aligned}
&\int_{B_{F^*}}\left|\left\langle\varphi, A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right\rangle\right|^s d\nu(\varphi)\\
&= \sum_{j=1}^{k}\left|\left\langle\tilde{\varphi}_j, A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right\rangle\right|^s \nu(\tilde{\varphi}_j)\\
&= \sum_{j=1}^{k}\left|\left\langle\frac{\varphi_j}{\|\varphi_j\|}, A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right\rangle\right|^s .\nu_j.\delta_j(\tilde{\varphi}_j)\\
&= \sum_{j=1}^{k}\left|\left\langle\frac{\varphi_j}{\|\varphi_j\|}, A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right\rangle\right|^s .\frac{\|\varphi_j\|^s}{\sum_{j=1}^k\|\varphi_j\|^s}\\
&= \frac{1}{\left\|(\varphi_j)_{j=1}^k\right\|_s^s}\sum_{j=1}^{k}\left|\left\langle\varphi_j, A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right\rangle\right|^s.
\end{aligned}$$

From the previous equalities and from Theorem 6.3 we have

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_n=1}^{m}\left(\sum_{j=1}^{k}\left|\left\langle\varphi_j, A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right\rangle\right|^s\right)^{\frac{q}{s}}\right)^{\frac{1}{q}}\\
&=\left(\sum_{j_1,\ldots,j_n=1}^{m}\left(\int_{B_{F^*}}\left|\left\langle\varphi, A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right\rangle\right|^s d\nu(\varphi)\right)^{\frac{q}{s}}\right)^{\frac{1}{q}}\left\|(\varphi_j)_{j=1}^k\right\|_s\\
&\leq\left\|\left(A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right)_{j\in\mathbb{N}_m^n}\right\|_{m,(s,q)}\left\|(\varphi_j)_{j=1}^k\right\|_s\\
&\leq\|A\|_{mx(s,q;p_1,\ldots,p_n)}\prod_{l=1}^{n}\left\|(x_i^{(l)})_{i=1}^m\right\|_{w,p_l}\left\|(\varphi_j)_{j=1}^k\right\|_s.
\end{aligned}$$

and we obtain (6.2) with $\inf\sigma\leq\|A\|_{mx(s,q;p_1,\ldots,p_n)}$.

Reciprocally, with the same idea and using (6.2), given $\nu=\sum_{i=1}^k\nu_i\delta_i$ a discrete probability measure onto B_{F^*} we obtain

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_n=1}^{m}\left(\int_{B_{F^*}}\left|\left\langle\varphi, A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right\rangle\right|^s d\nu(\varphi)\right)^{\frac{q}{s}}\right)^{\frac{1}{q}}\\
&=\left(\sum_{j_1,\ldots,j_n=1}^{m}\left(\sum_{j=1}^{k}\left|\left\langle\nu_j^{\frac{1}{s}}\varphi_j, A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right\rangle\right|^s\right)^{\frac{q}{s}}\right)^{\frac{1}{q}}\\
&\leq\sigma\prod_{l=1}^{n}\left\|(x_i^{(l)})_{i=1}^m\right\|_{w,p_l}\left\|(\nu_j^{\frac{1}{s}}\varphi_j)_{j=1}^k\right\|_s\\
&\leq\sigma\prod_{l=1}^{n}\left\|(x_i^{(l)})_{i=1}^m\right\|_{w,p_l}.
\end{aligned}$$

The previous inequality holds for every $\nu\in W(B_{F^*})$, since the discrete probability measures are dense in $W(B_{F^*})$. Therefore, from Theorem 6.3 we obtain

$$\left\|\left(A(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)})\right)_{j\in\mathbb{N}_m^n}\right\|_{mx(s,q)}\leq\sigma\prod_{l=1}^{n}\left\|(x_i^{(l)})_{i=1}^m\right\|_{w,p_l},$$

for all $m\in\mathbb{N}$ and

$$\|A\|_{mx(s,q;p_1,\ldots,p_n)}=\inf\sigma.$$

□

6.1. **A quotient theorem.** For linear operators, $S\in\mathcal{L}(E;F)$ is (s,p)-mixing summing if and only if TS is absolutely p-summing for all $T\in\Pi_s(F;G)$. In other words

$$\Pi_{mx(s,p)}(E;F)=(\Pi_s(F;G))^{-1}\circ\Pi_p(E;G).$$

For details we refer to [19, Section 32] and [65]. In this section we show that our approach provides a perfect multilinear extension of this result. We show that the following assertions are equivalent:

- $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ is multiple $(s, q; p_1, \ldots, p_n)$-mixing summing.
- $u \circ T \in \mathcal{L}_{mas(q;p_1,\ldots,p_n)}(E_1, \ldots, E_n; G)$ for all $u \in \Pi_s(F; G)$ and $T \in \mathcal{L}(E_1, \ldots, E_n; F)$.

Using a different notation, we will show the following quotient theorem:

$$\Pi_{mx(s,q;p_1,\ldots,p_n)}(E_1, \ldots, E_n; F) = (\Pi_s(F; G))^{-1} \circ \mathcal{L}_{mas(q;p_1,\ldots,p_n)}(E_1, \ldots, E_n; G) \tag{6.4}$$

for all $E_1, \ldots, E_n, F$ and G.

The quotient theorem (6.4) is a direct consequence of the forthcoming Propositions 6.6 and 6.7. First we need the following lemma:

Lemma 6.5. *Let $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ be so that*

$$u \circ A \in \mathcal{L}_{mas(p;p_1,\ldots,p_n)}(E_1, \ldots, E_n; G)$$

for all Banach space G and all $u \in \Pi_r(F; G)$. Then, there is a $C \geq 0$ such that

$$\|u \circ A\|_{(p;p_1,\ldots,p_n)} \leq C\pi_r(u). \tag{6.5}$$

Proof. Suppose that (6.5) is not true. So, for all positive integer k there exist Banach spaces F_k and $u_k \in \Pi_r(F; F_k)$ so that

$$\pi_r(u_k) \leq \frac{1}{2^k} \text{ and } \|u_k \circ A\|_{(p;p_1,\ldots,p_n)} \geq k.$$

Let $J_k : F_k \to \ell_2((F_k)_{k=1}^{\infty})$ and $Q_j : \ell_2((F_k)_{k=1}^{\infty}) \to F_j$ be the canonical maps for all positive integers j, k. Since

$$\pi_r\left(\sum_{k=n_1}^{n_2} J_k \circ u_k\right) \leq \sum_{k=n_1}^{n_2} \pi_r(J_k \circ u_k) \leq \sum_{k=n_1}^{n_2} \pi_r(u_k) \leq \sum_{k=n_1}^{n_2} \frac{1}{2^k}$$

it follows that

$$u := \sum_{j=1}^{\infty} J_j \circ u_j \in \Pi_r(F; \ell_2((F_k)_{k=1}^{\infty})).$$

Since $u_k = Q_k \circ u$, we thus have

$$k \leq \|u_k \circ A\|_{(p;p_1,\ldots,p_n)} = \|Q_k \circ u \circ A\|_{(p;p_1,\ldots,p_n)} \leq \|u \circ A\|_{(p;p_1,\ldots,p_n)},$$

a contradiction. □

Proposition 6.6. *If $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is so that $u \circ A \in \mathcal{L}_{mas(q;p_1,\ldots.,p_n)}(E_1, \ldots, E_n; G)$ for all $u \in \Pi_s(F; G)$, then*

$$A \in \Pi_{mx(s,q;p_1,\ldots.,p_n)}(E_1, \ldots, E_n; F).$$

Proof. Let $x_i^{(j)} \in E_j$ with $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$. Consider $S : F \to \ell_s^k$ defined by

$$S(y) = (\varphi_j(y))_{j=1}^{k}.$$

It is not difficult to show that

$$\pi_s(S) \leq \left\|(\varphi_j)_{j=1}^{k}\right\|_s.$$

Since $S\circ A\in\mathcal{L}_{mas(q;p_1,\dots,p_n)}(E_1,\dots,E_n;\ell_s^k)$ and invoking Lemma 6.5, there is a constant $C>0$ so that

$$\begin{aligned}
&\left(\sum_{j_1,\dots,j_n=1}^{m}\left(\sum_{j=1}^{k}\left|\left\langle\varphi_j,A(x_{j_1}^{(1)},\dots,x_{j_n}^{(n)})\right\rangle\right|^s\right)^{\frac{q}{s}}\right)^{\frac{1}{q}}\\
&=\left(\sum_{j_1,\dots,j_n=1}^{m}\left\|S\circ A\left(x_{j_1}^{(1)},\dots,x_{j_n}^{(n)}\right)\right\|_s^q\right)^{\frac{1}{q}}\\
&\leq\|S\circ A\|_{(q;p_1,\dots,p_n)}\prod_{j=1}^{n}\left\|\left(x_i^{(j)}\right)_{i=1}^{m}\right\|_{w,p_j}\\
&\leq C\pi_s(S)\prod_{j=1}^{n}\left\|\left(x_i^{(j)}\right)_{i=1}^{m}\right\|_{w,p_j}\\
&\leq C\left\|(\varphi_j)_{j=1}^{k}\right\|_s\prod_{j=1}^{n}\left\|\left(x_i^{(j)}\right)_{i=1}^{m}\right\|_{w,p_j}.
\end{aligned}$$

□

Proposition 6.7. *If $A\in\Pi_{mx(s,q;p_1,\dots,p_n)}(E_1,\dots,E_n;F)$, then*

$$u\circ A\in\Pi_{(q;p_1,\dots,p_n)}(E_1,\dots,E_n;G)\tag{6.6}$$

and

$$\|u\circ A\|_{(q;p_1,\dots,p_n)}\leq\pi_s(u)\,\|A\|_{mx(s,q;p_1,\dots,p_n)}\tag{6.7}$$

for all $u\in\Pi_s(F;G)$.

Proof. Let $x_i^{(j)}\in E_j$ with $(i,j)\in\{1,\dots,m\}\times\{1,\dots,n\}$. Given $\varepsilon>0$ there are $\tau_{j_1,\dots,j_n}\in K$ and $y_{j_1,\dots,j_n}\in F$ so that

$$A(x_{j_1}^{(1)},\dots,x_{j_n}^{(n)})=\tau_{j_1,\dots,j_n}y_{j_1,\dots,j_n}$$

and

$$\begin{aligned}
&\left\|(\tau_{j_1,\dots,j_n})_{j_1,\dots,j_n=1}^{m}\right\|_r\left\|(y_{j_1,\dots,j_n})_{j_1,\dots,j_n=1}^{m}\right\|_{w,s}\\
&<(1+\varepsilon)\left\|\left(A(x_{j_1}^{(1)},\dots,x_{j_n}^{(n)})\right)_{j_1,\dots,j_n=1}^{m}\right\|_{mx(s,q))}\\
&\leq(1+\varepsilon)\|A\|_{mx(s,q;p_1,\dots,p_n)}\prod_{j=1}^{n}\left\|\left(x_i^{(j)}\right)_{i=1}^{m}\right\|_{w,p_j}.
\end{aligned}$$

Hence, using Hölder's Inequality we obtain

$$\begin{aligned}
&\left\| \left(u \circ A(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)}) \right)_{j_1,\ldots,j_n=1}^{m} \right\|_q \\
&\leq \left\| (\tau_{j_1,\ldots,j_n})_{j_1,\ldots,j_n=1}^{m} \right\|_r \left\| (u\,(y_{j_1,\ldots,j_n}))_{j_1,\ldots,j_n=1}^{m} \right\|_s \\
&\leq (1+\varepsilon)\, \pi_s\,(u)\, \|A\|_{mx(s,q;p_1,\ldots,p_n)} \prod_{j=1}^{n} \left\| \left(x_i^{(j)} \right)_{i=1}^{m} \right\|_{w,p_j}
\end{aligned}$$

and making $\varepsilon \to 0$ we get (6.6) and (6.7). □

6.2. **Coherence and compatibility.** The polynomial version of multiple mixing summing operators can be stated by using the symmetric multilinear operator associated to the polynomials:

Definition 6.8. *Let $0 < p \leq s < \infty$. A polynomial $P \in \mathcal{P}(^nE; F)$ is multiple (s,p)-mixing summing if $\check{P}$ is multiple $(s,p;p)$-mixing summing. Besides,*

$$\|P\|_{mx(s,p)} := \left\|\check{P}\right\|_{mx(s,p;p)}.$$

The following proposition, whose proof is standard, shows that, as it happens to multiple summing multilinear operators, coincidence results for multiple mixing summing multilinear operators imply in coincidence results for smaller degrees:

Proposition 6.9. *If $\mathcal{L}(E_1,\ldots,E_n;F) = \Pi_{mx(s,q;p_1,\ldots,p_n)}(E_1,\ldots,E_n;F)$ then*

$$\mathcal{L}(E_{k_1},\ldots,E_{k_j};F) = \Pi_{mx(s,q;p_{k_1},\ldots,p_{k_j})}(E_{k_1},\ldots,E_{k_j};F)$$

whenever $1 \leq j < n$ and $\{k_1 < \cdots < k_j\} \subset \{1,\ldots,n\}$.

Similarly to the previous section one can show that $\left(\mathcal{P}^n_{mx(s,p)}, \|\cdot\|_{mx(s,p)}\right)_{n=1}^{\infty}$ is coherent and for each n it is compatible with the operator ideal $\left(\Pi_{mx(s,p)}, \pi_{mx(s,p)}\right)$. For example we prove (i) of Definition 2.2:

Proposition 6.10. *If $P \in \mathcal{P}_{mx(s,p)}(^nE;F)$ and $a \in E$, then $P_a \in \mathcal{P}_{mx(s,p)}(^{n-1}E;F)$ and*

$$\|P_a\|_{mx(s,p)} \leq \|P\|_{mx(s,p)}\,\|a\|.$$

Proof. Since $\check{P} \in \Pi_{mx(s,p)}(^nE;F)$ we have

$$\left(\sum_{j_1,\ldots,j_n=1}^{m} \left(\sum_{j=1}^{k} \left| \left\langle \varphi_j, \check{P}(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)}) \right\rangle \right|^s \right)^{\frac{p}{s}} \right)^{\frac{1}{p}} \leq \sigma \prod_{l=1}^{n} \left\| (x_i^{(l)})_{i=1}^{m} \right\|_{w,p} \left\| (\varphi_j)_{j=1}^{k} \right\|_s.$$

and by choosing $x_1^{(n)} = a$ and $x_j^{(n)} = 0$ for $j > 1$ we have

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_{n-1}=1}^{m} \left(\sum_{j=1}^{k} \left| \left\langle \varphi_j, \check{P}_a(x_{j_1}^{(1)}, \ldots, x_{j_{n-1}}^{(n-1)}) \right\rangle \right|^s \right)^{\frac{p}{s}} \right)^{\frac{1}{p}} \\
&= \left(\sum_{j_1,\ldots,j_{n-1}=1}^{m} \left(\sum_{j=1}^{k} \left| \left\langle \varphi_j, \check{P}(x_{j_1}^{(1)}, \ldots, x_{j_{n-1}}^{(n-1)}, a) \right\rangle \right|^s \right)^{\frac{p}{s}} \right)^{\frac{1}{p}} \\
&= \left(\sum_{j_1,\ldots,j_n=1}^{m} \left(\sum_{j=1}^{k} \left| \left\langle \varphi_j, \check{P}(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)}) \right\rangle \right|^s \right)^{\frac{p}{s}} \right)^{\frac{1}{p}} \\
&\leq \|P\|_{mx(s,p)} \|a\| \prod_{l=1}^{n-1} \left\| (x_i^{(l)})_{i=1}^m \right\|_{w,p} \left\| (\varphi_j)_{j=1}^k \right\|_s .
\end{aligned}$$

□

7. Final comments and directions for further research

The concepts of multiple mixing summing and multiple $(p;q;r_1,\ldots,r_n)$-summing polynomials/multilinear operators, as natural extensions of the notion of multiple summing multilinear operators, can be further investigated following different directions: coincidence theorems, generalizations to holomorphic mappings, or inclusion theorems, among others.

The study of coincidence theorems may follow the lines of [10] combined with the results from the respective linear theories; the study of holomorphic mappings may follow [37] and for inclusion theorems [60] is certainly a good source of inspiration.

We encourage the interested reader to investigate other variants of mixing summability and $(p;q;r_1,\ldots,r_n)$-summability following the lines given in [5, 43, 45, 54].

References

[1] D. Achour, Multilinear extensions of absolutely $(p;q;r)$-summing operators, Rend. Circ. Mat. Palermo (in press). DOI: 10.1007/s12215-011-0054-2.

[2] R. Alencar and M. C. Matos, Some classes of multilinear mappings between Banach spaces, Publicaciones del Departamento de Análisis Matemático 12, Universidad Complutense Madrid, (1989), 1-34.

[3] N. Alon, K. Makarychev, Y. Makarychev, and A. Naor, Quadratic forms on graphs, Invent. Math. **163** (2006), 499–522.

[4] S. Banach, Théorie des opérations linéaires, PWN, 1932.

[5] J. Barbosa, G. Botelho, D. Diniz and D. Pellegrino, Spaces of absolutely summing polynomials, Math. Scand. **101** (2007), 219-237.

[6] H.F. Bohnenblust and E. Hille, On the absolute convergence of Dirichlet series, Ann. of Math. (2) **32** (1931), 600-622.

[7] G. Botelho, H.-A. Braunss, H. Junek and D. Pellegrino, Holomorphy types and ideals of multilinear mappings, Studia Math. **177** (2006), 43-65.

[8] G. Botelho, H.-A. Braunss, H. Junek and D. Pellegrino, Inclusions and coincidences for multiple summing multilinear mappings, Proc. Amer. Math. Soc. **137** (2009), 991-1000.

[9] G. Botelho and D. Pellegrino, Coincidence situations for absolutely summig non-linear mappings, Port. Math. **64** (2007), 175-191.

[10] G. Botelho and D. Pellegrino, When every multilinear mapping is multiple summing, Math. Nachr. **282** (2009), 1414-1422.

[11] G. Botelho, D. Pellegrino and P. Rueda, Cotype and absolutely summing linear operators, Math. Z. **267** (2011), 1–7.

[12] M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor, The Grothendieck constant is strictly smaller than Krivine's bound (2011). Available at arXiv:1103.6161v2.

[13] J.W. Calkin, Two sided ideals and congruences in the ring of bounded operators in Hilbert space, Ann. of Math. (2) **42** (1941), 839–873.

[14] D. Carando, V. Dimant and S. Muro, Coherent sequences of polinomial ideals on Banach spaces. Math. Nachr. **282** (2009), 1111-1133.

[15] B. Carl and A. Defant, Tensor products and Grothendieck type inequalities of operators in L_p-spaces, Trans. Amer. Math. Soc. **331** (1992), 55–76.

[16] J.A. Chávez-Domínguez, Lipschitz $(q;p)$-mixing operators, Proc. Amer. Math. Soc. (in press).

[17] D. Chen and B. Zheng, Remarks on Lipschitz p-summing operators, Proc. Amer. Math. Soc. (in press).

[18] Y. S. Choi, S. G. Kim, Y. Meléndez and A. Tonge. Estimates for absolutely summing norms of polynomials and multilinear maps, Quarterly J. Math. **52** (2001), 1-12.

[19] A. Defant and K. Floret, Tensor norms and operator ideals, North-Holland Mathematics Studies 176, North Holland, Amsterdam, 1992.

[20] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes and K. Seip, The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive, Ann. of Math. (2) **174** (2011), 485-497.

[21] A. Defant, D. Popa and U. Schwarting, Coordenatewise multiple summing operators on Banach spaces, J. Funct. Anal. **259** (2010), 220–242.

[22] A. Defant and P. Sevilla-Peris, A new multilinear insight on Littlewood's 4/3-inequality, J. Funct. Anal. **256** (2009), 1642-1664.

[23] J. Diestel, An elementary characterization of absolutely summing operators, Math. Ann. **196** (1972), 101–105.

[24] J. Diestel, J. H. Fourie and J. Swart, The metric theory of tensor products, Grothendieck's Résumé revisited, American Mathematical Society, Providence, 2008.

[25] J. Diestel, H. Jarchow and A. Pietsch, Operator ideals, in Handbook of the geometry of Banach spaces, Vol. I, 437-496, North-Holland, Amsterdam, 2001.

[26] J. Diestel, A.M. Peralta and D. Puglisi, Sequential w-right continuity and summing operators, Math. Nachr. **284** (2011), 664–680.

[27] J. Diestel, H. Jarchow and A. Tonge, Absolutely summing operators, Cambridge University Press, 1995.

[28] V. Dimant, Strongly -summing multilinear operators, J. Math. Anal. Appl. **278** (2003), 182-193.

[29] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag, London, 1999.

[30] D. Diniz, G.A. Muñoz-Fernandez, D. Pellegrino and J.B. Seoane-Sepúlveda, The asymptotic growth of the constants in the Bohnenblust-Hille inequality is optimal, arXiv:1108.1550.

[31] A. Dvoretzky and C.A. Rogers, Absolute and unconditional convergence in normed linear spaces, Proc. Nat. Acad. Sci. USA **36** (1950), 192-197.

[32] P.C. Fishburn and J.A. Reeds, Bell inequalities, Grothendieck's constant, and root two, SIAM J. Discrete Math. **7** (1994), 48–56.

[33] A. Grothendieck, Sur les applications linéaires faiblement compactes d´espaces du type $C(K)$, Canad. J. Math. **5** (1953), 129–173.

[34] A. Grothendieck, Résumé de la théorie metrique des produits tensoriels topologiques, Bol. Soc. Mat. São Paulo **8** (1953/1956), 1-79.

[35] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Memoirs Acad. Math. Soc. **16**, 1955.

[36] A. Hinrichs and A. Pietsch, p-nuclear operators in the sense of Grothendieck, Math. Nachr. **283** (2010), 232–261.

[37] H. Junek, M.C. Matos and D. Pellegrino, Inclusion theorems for absolutely summing holomorphic mappings. Proc. Amer. Math. Soc. **136** (2008), 3983–3991.

[38] J.-T. Lapresté, Opérateus sommants et factirisations à travers les espaces L^p, Studia Math. **57** (1976), 47–83.

[39] A. Lima, V. Lima, E. Oja, Absolutely summing operators on $C[0,1]$ as a tree space and the bounded approximation property, J. Funct. Anal. **259** (2010) 2886–2901

[40] J. Lindenstrauss and A. Pełczyński, Absolutely summing operators in Lp spaces and their applications, Studia Math. **29** (1968), 275–326.

[41] J. Littlewood, On bounded bilinear forms in an infinite number of variables, Q. J. Math. **2** (1930), 167–171.

[42] M.S. Macphail, Absolute and unconditional convergence, Bull. Amer. Math. Soc. **53** (1947), 121–123.

[43] M.C. Matos, Nonlinear absolutely summing multilinear mappings between Banach spaces, Math. Nachr. **258** (2003), 71–89.

[44] M. C. Matos, Fully absolutely summing and Hilbert-Schmidt multilinear mapping. Collect. Math. **54** (2003), 111–136.

[45] M. C. Matos and D. Pellegrino, Fully summing mappings between Banach spaces, Studia Math. **178** (2007), 47–61.

[46] R.D. Mauldin, The Scottish book, Birkhauser, 1982.

[47] B. Maurey, Théorèmes de factorisation pour les opérateurs à valeurs dans les espaces L^p, Astérisque 11, Paris, 1974.

[48] B. Maurey and G. Pisier, Séries de variables aleatoires vectorielles indépendantes et proprietés géometriques des espaces de Banach, Studia Math. **58** (1976), 45-90.

[49] B. Mitiagin and A. Pełczyński, Nuclear operators and approximative dimensions, Proceedings International Congress of Mathematics, Moscow 1966.

[50] J. Mujica, Complex analysis in Banach spaces, North-Holland Mathematics Studies 120, North-Holland, 1986.

[51] G.A. Muñoz-Fernandez, D. Pellegrino and J.B. Seoane-Sepúlveda, Estimates for the asymptotic behavior of the constants in the Bohnenblust-Hille inequality, Linear Multilinear Algebra (in press).

[52] L. Nachbin, Topology on Spaces of Holomorphic Mappings, Springer, New York, 1969.

[53] S. Okada, W.J. Ricker and L. Rodríguez-Piazza, Absolutely summing convolution operators in $L^p(G)$, Proc. London Math. Soc. **102** (2011), 843–882.

[54] D. Pellegrino, Almost summing mappings, Arch. Math. **82** (2004), 68–80.

[55] D. Pellegrino and J. Ribeiro, On almost summing polynomials and multilinear mappings, Linear Multilinear Algebra (in press). DOI: 10.1080/03081087.2011.603315.

[56] D. Pellegrino and J. Santos, Absolutely summing multilinear operators: a panorama, Quaest. Math. **34** (2011), 447–478.

[57] D. Pellegrino, J. Santos and J.B. Seoane-Sepúlveda, Some techniques on nonlinear analysis and applications. Adv. Math. **229** (2012), 1235–1265.

[58] D. Pellegrino and J.B. Seoane-Sepúlveda, New upper bounds for the constants in the Bohnenblust-Hille inequality, J. Math. Anal. Appl. **386** (2012), 300–307.

[59] D. Pérez-García, Operadores multilineales absolutamente sumantes, Ph.D. thesis, Universidad Complutense de Madrid, 2003.

[60] D. Pérez-García, The inclusion theorem for multiple summing operators, Studia Math. **165** (2004), 275–290.

[61] D. Pérez-García, Comparing different classes of absolutely summing multilinear operators, Arch. Math. **85** (2005), 258–267.

[62] D. Pérez-García and I. Villanueva, Multiple summing operators on Banach spaces, J. Math. Anal. Appl. **285** (2003), 86–96.

[63] A. Pietsch, Absolut p-summierende Abbildungen in normieten Raumen, Studia Math. **27** (1967), 333–353.

[64] A. Pietsch, Theorie der Operatorenideale, Friedrich-Schiller-Universität, Jena, 1972.

[65] A. Pietsch, Operator Ideals, Deutscher Verlag der Wiss, 1978 and North Holland, Amsterdam, 1980.

[66] A. Pietsch, Ideals of multilinear functionals, Proceedings of the Second International Conference on Operator Algebras, Ideals and Their Applications in Theoretical Physics, Teubner-texte Math. 67 (Teubner, Leipzig, 1983) pp. 185–199.

[67] A. Pietsch, History of Banach spaces and linear operators, Birkhauser, 2007.

[68] D. Popa, Reverse inclusions for multiple summing operators, J. Math. Anal. Appl. **350** (2009), 360–368.

[69] D. Popa, A new distinguishing feature for summing, versus dominated and multiple summing operators, Arch. Math. **96** (2011), 455–462.

[70] J. Puhl, Quotienten von Operatorenidealen, Math. Nachr. **79** (1977), 131–144.

[71] M.S. Ramanujan and E. Schock, Operator ideals and spaces of bilinear operators, Linear Multilinear Algebra **18** (1985), 307–318.

[72] C.A.S. Soares, Aplicações multilineares e polinômios misto somantes, PhD Thesis, Universidade Estadual de Campinas (UNICAMP), 1998.

[73] M. L. V. Souza, Aplicações multilineares completamente absolutamente somantes, PhD Thesis, Universidade Estadual de Campinas (UNICAMP), 2003.

[74] M. Talagrand, Cotype and $(q, 1)$-summing norms in Banach spaces, Invent. Math. **110** (1992), 545–556.

[75] H. Weyl, Inequalities between the two kinds of eigenvalues of a linear transformation, Proc. Acad. Sci. USA **35** (1949), 408–411.

Centro de Ensino Superior do Seridó,
Universidade Federal do Rio Grande do Norte,
Rua Joaquim Gregório, S/N - Penedo,
Caicó, 59300-000, Brazil.
E-mail address: thiagobernardino@yahoo.com.br

Departamento de Matemática,
Universidade Federal da Paraíba,
58.051-900 - João Pessoa, Brazil.
E-mail address: pellegrino@pq.cnpq.br

Departamento de Análisis Matemático,
Facultad de Ciencias Matemáticas,
Plaza de Ciencias 3,
Universidad Complutense de Madrid,
Madrid, 28040, Spain.
E-mail address: jseoane@mat.ucm.es

Departamento de Matemática/ICENE,
UFTM - Universidade Federal do Triângulo Mineiro,
Rua Getúlio Guaritá, 159,
CEP 38.025-440 - Uberaba-MG, Brazil.
E-mail address: marcelalvsouza@gmail.com